1. Gauss-Bonnet

This week we will talk about the Gauss-Bonnet theorem for general orientable compact surfaces with boundary.

Theorem 1.1. *(The general statement of Gauss-Bonnet) Let Σ be an orientable compact surfaces with boundary, then $\int KdA + \int k_g ds = 2\pi\chi(\Sigma)$. Where $\chi(\Sigma) = 2 - 2g - k$, k denotes number of boundary components(by taking out k disks from a closed genus g surface which has $\chi = 2 - 2g$).*

Remark 1.2. *Since $\chi(\Sigma) = v - e + f$ by triangulation and it is an invariant for the surface(From algebraic topology). In fact by Euler theorem, we know a sphere is of characteristic 2. We can compute explicitly χ for the disk and annulus and show that $\chi(\Sigma) = 2 - 2g - k$.*

Topological applications:
(1) $(K > 0 and k_g >= 0) or (K >= 0 and k_g > 0)$ implies topologically a disk.
(2) Demonstrate that a flat cylinder has $K = 0$ and $k_g = 0$, contrast it with the non-existence of closed surfaces in R^3 with $K = 0$.
(3) If M is a compact surface in $\mathbb{R}^3$ with no boundary and have genus bigger than 1, then M will have points where the Gauss curvature are respectively positive, negative and 0.
(4) A torus has Euler characteristic 0, so its total curvature must also be zero. If the torus carries the ordinary Riemannian metric from its embedding in $\mathbb{R}^3$, then the inside has negative Gaussian curvature, the outside has positive Gaussian curvature, and the total curvature is indeed 0. It is also possible to construct a torus by identifying opposite sides of a square, in which case the Riemannian metric on the torus is flat and has constant curvature 0, again resulting in total curvature 0. It is not possible to specify a Riemannian metric on the torus with everywhere positive or everywhere negative Gaussian curvature.
(5)The theorem also has interesting consequences for triangles. Suppose M is some 2-dimensional Riemannian manifold (not necessarily compact), and we specify a "triangle" on M formed by three geodesics. Then we can apply GaussBonnet to the surface T formed by the inside of that triangle and the piecewise boundary given by the triangle itself. The geodesic curvature of geodesics being zero, and the Euler characteristic of T being 1, the theorem then states that the sum of the turning angles of the geodesic triangle is equal to 2π minus the total curvature within the triangle. Since the turning angle at a corner is equal to π minus the interior angle, we can rephrase this as follows: The sum of interior angles of a geodesic triangle is equal to π plus the total curvature enclosed by the triangle. In the case of the plane (where the Gaussian curvature is 0 and geodesics are straight lines), we recover the familiar formula for the sum of angles in an ordinary triangle. On the standard sphere, where the curvature is everywhere 1, we see that the angle sum of geodesic triangles is always bigger than π. On a hyperbolic surface, we will have the angle sum of geodesic triangles is always smaller than π. Summary:

$$\int KdA = (\alpha_1 + \alpha_2 + \alpha_3) - \pi. \tag{1.1}$$

2. Some Notes:

- Thm 2: Every compact surface has a triangulation.
  (A short proof can be found in Doyle & Moran 1968.)

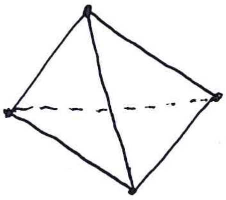
- ~~As~~ By parametrization $f: (a,b) \times \mathbb{R} \to \mathbb{R}^3$

$$f(t, \varphi) = (x(t)\cos\varphi,\ x(t)\sin\varphi,\ z(t))$$

Let $x(t) = 1$, $z(t) = t$, this is the cylinder,
we have $K = -\frac{x''}{x} = 0$. (by hw 5). and $K_g = 0$ for $t =$ const

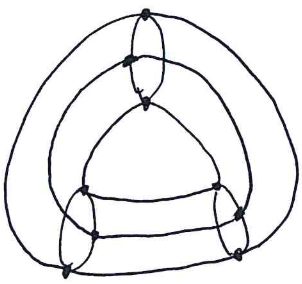
- The triangulations and retangulations.

· Disk (Tetrahedron):

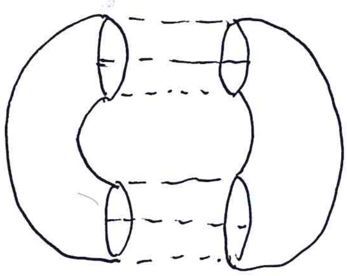


$$\chi = 4 - 6 + 4 = 2$$

· Torus:



$$\chi = 9 - 18 + 9 = 0.$$



$$\chi = 2 + 2 + (-2 + 3 - 3) \times 2 = 0.$$